Complex analysis 3

Frames 1 to 38

Learning outcomes

When you have completed this Programme you will be able to:

- Expand a function of a complex variable about the origin in a Maclaurin series
- Determine the circle and radius of convergence of a Maclaurin series expansion
- Recognise singular points in the form of poles of order n, removable and essential singularities
- Expand a function of a complex variable about a point in the complex plane in a Taylor series, transforming the coordinates with a shift of origin
- Expand a function of a complex variable about a singular point in a Laurent series
- Recognise the principal and analytic parts of the Laurent series and link the form of the principal part to the type of singularity
- Recognise the residue of a Laurent series and state the Residue theorem
- Calculate the residues at the poles of an expression without resort to deriving the Laurent series
- Evaluate certain types of real integrals using the Residue theorem

Maclaurin series

1

You will recall that the Maclaurin series expansion of the function of a real variable x with output $f(x)$ is given as

$$f(x) = f(0) + xf'(0) + x^2\frac{f''(0)}{2!} + x^3\frac{f'''(0)}{3!} + \cdots + x^n\frac{f^{(n)}(0)}{n!} + \cdots$$

This is an infinite series expansion of $f(x)$ about the point $x = 0$. Because the series on the right-hand side of this equation contains an infinite number of terms, the right-hand side may only converge for a restricted set of values of x. Consequently, this expansion is only valid for that restricted set of values. For example, the expression $f(x) = (1 - x)^{-1}$ has the Maclaurin series expansion ....

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$$\boxed{f(x) = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots}$$

Because

$$\begin{aligned}
&f(x) = (1-x)^{-1} \text{ and so } f(0) = (1-0)^{-1} = 1\\
&f'(x) = (1-x)^{-2} \text{ and so } f'(0) = (1-0)^{-2} = 1\\
&f''(x) = 2(1-x)^{-3} \text{ and so } f''(0) = 2(1-0)^{-3} = 2\\
&f'''(x) = 3!(1-x)^{-4} \text{ and so } f'''(0) = 3!(1-0)^{-4} = 3!\\
&\qquad\vdots \qquad\qquad\qquad\qquad \vdots\\
&f^{(n)}(x) = n!(1-x)^{-(n+1)} \text{ and so } f^{(n)}(0) = n!(1-0)^{-(n+1)} = n!
\end{aligned}$$

Therefore, substituting into the Maclaurin series expansion, we find

$$\begin{aligned}
f(x) &= f(0) + xf'(0) + x^2\frac{f''(0)}{2!} + x^3\frac{f'''(0)}{3!} + \cdots + x^n\frac{f^{(n)}(0)}{n!} + \cdots\\
&= 1 + x \times 1 + x^2 \times \frac{2!}{2!} + x^3 \times \frac{3!}{3!} + \cdots + x^n \times \frac{n!}{n!} + \cdots\\
&= 1 + x + x^2 + x^3 + \cdots + x^n + \cdots
\end{aligned}$$

This same result could also be derived by using the binomial theorem or even by performing the long division of 1 by $1 - x$. However, performing the algorithmic procedure is one thing, but knowing that the result of the procedure is valid is another. To determine the validity of the expansion we resort to convergence tests, and in this case we use the ratio test. To refresh your memory, the ratio test for the infinite series

$$f(x) = a_0(x) + a_1(x) + a_2(x) + a_3(x) + \cdots + a_n(x) + \cdots$$

is that given

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = L \text{ then if } \begin{array}{l} L < 1 \text{ the series converges}\\ L > 1 \text{ the series diverges}\\ L = 1 \text{ the test fails and an alternative}\\ \qquad \text{convergence test is required.}\end{array}$$

▶

Applying the ratio test to the Maclaurin series expansion

$$f(x) = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

tells us that

The series converges for ....
The series diverges for ....
The test fails for ....

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The series converges for $-1 < x < 1$
The series diverges for $x < -1$ or $x > 1$
The test fails for $x = \pm 1$

Because

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right| = \lim_{n\to\infty}|x| = |x|, \text{ so}$$

if $|x| < 1$, that is $-1 < x < 1$, the series converges and so the expansion is valid
$|x| > 1$, that is $x < -1$ or $x > 1$, the series diverges and so the expansion is invalid
$|x| = 1$, that is $x = \pm 1$, the ratio test fails to give a conclusion.

By inspection, when $x = 1$ the series clearly diverges and when $x = -1$ the sum of terms alternates between 1 and 0 as each successive term is added. Clearly the series does not converge and so, therefore, it must diverge when $x = -1$.

Everything that has been said about the Maclaurin series expansion of an expression involving a real variable x can equally be said about an expression involving a complex variable z. That is, if $f(z)$ is a function in the complex variable z, analytic at $z = 0$, then the Maclaurin series expansion is

$$f(z) = f(0) + zf'(0) + z^2\frac{f''(0)}{2!} + z^3\frac{f'''(0)}{3!} + \cdots$$

So, the Maclaurin series expansion of $f(z) = \sin z$ is ....

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$$\boxed{f(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + \frac{(-1)^n z^{2n+1}}{(2n+1)!} + \cdots}$$

Because

$f(z) = \sin z$ and so $f(0) = \sin 0 = 0$

$f'(z) = \cos z$ and so $f'(0) = \cos 0 = 1$

$f''(z) = -\sin z$ and so $f''(0) = -\sin 0 = 0$

$f'''(z) = -\cos z$ and so $f'''(0) = -\cos 0 = -1$

$\vdots \qquad\qquad\qquad \vdots$

Therefore

$$\begin{aligned} f(z) &= f(0) + zf'(0) + z^2\frac{f''(0)}{2!} + z^3\frac{f'''(0)}{3!} + \cdots \\ &= 0 + z \times 1 + z^2 \times \frac{0}{2!} + z^3 \times \frac{(-1)}{3!} + \cdots \\ &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + \frac{(-1)^n z^{2n+1}}{(2n+1)!} + \cdots \end{aligned}$$

Furthermore, applying the ratio test tells us that this series expansion is valid for ....

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$$\boxed{\text{all finite values of } z}$$

Because

$$\begin{aligned} \underset{n\to\infty}{Lim}\left|\frac{a_{n+1}(z)}{a_n(z)}\right| &= \underset{n\to\infty}{Lim}\left|\frac{(-1)^{n+1} z^{2(n+1)+1}/[2(n+1)+1]!}{(-1)^n z^{2n+1}/[2n+1]!}\right| \\ &= \underset{n\to\infty}{Lim}\left|\frac{z^2}{(2n+3)(2n+2)}\right| = 0 < 1 \end{aligned}$$

So the expansion is valid for all finite values of z.

Try this one. The Maclaurin series expansion of $f(z) = \ln(1+z)$ is

$$\ln(1+z) = \ldots\ldots\ldots\ldots$$

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$$\boxed{\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots + \frac{(-1)^{n+1}z^n}{n} + \cdots \quad n = 1, 2, \ldots}$$

Because

$f(z) = \ln(1+z)$ and so $f(0) = (1+0) = 0$

$f'(z) = (1+z)^{-1}$ and so $f'(0) = (1+0)^{-1} = 1$

$f''(z) = -(1+z)^{-2}$ and so $f''(0) = -(1+0)^{-2} = -1$

$f'''(z) = 2(1+z)^{-3}$ and so $f'''(0) = 2(1+0)^{-3} = 2$

$f^{(iv)}(z) = -3!(1+z)^{-4}$ and so $f^{(iv)}(0) = -3!(1+0)^{-4} = -3!$

$\vdots \qquad \vdots$

$f^{(n)}(z) = (-1)^{n+1}n!(1+z)^{-n}$ and so $f^{(n)}(0) = (-1)^{n+1}n!(1+0)^{-n}$

$= (-1)^{n+1}n!$

Therefore

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots + \frac{(-1)^{n+1}z^n}{n} + \cdots$$

This series is valid for

....

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$$\boxed{|z| < 1}$$

Because

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(z)}{a_n(z)}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}z^{n+1}/[n+1]}{(-1)^{n+1}z^n/[n]}\right| = \lim_{n\to\infty}\left|\frac{nz}{n+1}\right| = |z|$$

So if $|z| < 1$ the series converges and so the expansion is valid

$|z| > 1$ the series diverges and so the expansion is invalid

$|z| = 1$ the ratio test fails

We shall look at the case $|z| = 1$ a little later.

Move to the next frame

Radius of convergence

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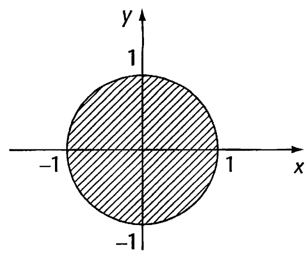
We have just seen that the Maclaurin expansion of $\ln(1+z)$ is valid for $|z| < 1$. This inequality defines the interior of a circle of radius 1 centred on the origin, namely $z = 1e^{j\theta}$.



This means that the expansion is valid for all z-values lying within this circle. The radius of the circle within which a series expansion is valid is called the *radius of convergence* of the series and the circle is called the *circle of convergence*.

Example

To find the infinite series expansion and radius of convergence of the expression $f(z) = \dfrac{z}{(1-3z)^2}$, we progress in stages, noting that $\dfrac{z}{(1-3z)^2} = z(1-3z)^{-2}$. We expand $(1-3z)^{-2}$ first.

By the binomial theorem, the expansion of $(1-3z)^{-2}$ is

$$(1-3z)^{-2} = \dots\dots\dots\dots$$

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$$\boxed{(1-3z)^{-2} = 1 + 6z + 27z^2 + 108z^3 + 405z^4 + \cdots}$$

Because

$$\begin{aligned}(1-3z)^{-2} &= \left(1 + (-2)\times(-3z) + \frac{(-2)(-3)\times(-3z)^2}{2!}\right.\\ &\qquad\left. + \frac{(-2)(-3)(-4)\times(-3z)^3}{3!} + \dots\right)\\ &= \left(1 + 6z + 3(-3z)^2 - 4(-3z)^3 + 5(-3z)^4 + \dots\right.\\ &\qquad\left. + (-1)^n(n+1)(-3)^n z^n + \dots\right)\\ &= 1 + 6z + 27z^2 + 108z^3 + 405z^4 + \dots + (n+1)3^n z^n + \dots\end{aligned}$$

and so

$$z(1-3z)^{-2} = z + 6z^2 + 27z^3 + 108z^4 + 405z^5 + \dots + (n+1)z^n z^{n+1} + \dots$$

The radius of convergence is then

$$\dots\dots\dots\dots$$

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$\boxed{1/3}$

Because

The general term of the expansion is $a_n(z) = (n+1)3^n z^{n+1}$ and so the ratio test tells us that

$$\underset{n\to\infty}{Lim}\left|\frac{a_{n+1}(z)}{a_n(z)}\right| = \underset{n\to\infty}{Lim}\left|\frac{(n+2)3^{n+1}z^{n+2}}{(n+1)3^n z^{n+1}}\right| = \underset{n\to\infty}{Lim}\left|\frac{3(n+2)z}{(n+1)}\right| = |3z|$$

So, if $|3z| < 1$, that is $|z| < 1/3$, then the series converges and the expansion is valid. The radius of convergence is therefore $1/3$.

Move to the next frame

Singular points

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Any point at which $f(z)$ fails to be analytic, that is where the derivative does not exist, is called a *singular point* (also called a singularity). For example

$$f(z) = \frac{1}{z-1}$$

is analytic everywhere in the finite complex plane except at the point $z = 1$ where not only is the derivative $f'(z)$ not defined but neither is $f(z)$. Accordingly, the point $z = 1$ is a singular point. There are different types of singular points, for now we shall look at just two of them.

Poles

If $f(z)$ has a singular point at z_0 and for some natural number n, $\underset{z\to z_0}{Lim}\{(z - z_0)^n f(z)\} = L \neq 0$ then the singular point is called a *pole of order n*. For example

$$f(z) = \frac{2z}{(z+4)^2}$$

has a singular point at $z = -4$ and because

$$\underset{z\to -4}{Lim}\left\{(z+4)^2 f(z)\right\} = \underset{z\to -4}{Lim}\{2z\} = -8 \neq 0$$

the singularity is a *pole of order* 2 (also called a *double pole*).

▶

Removable singularities

If $f(z)$ has a singular point at z_0 and $\underset{z \to z_0}{Lim}\{f(z)\}$ exists then the singular point is called a *removable singularity*. For example

$$f(z) = \frac{\sin z}{z}$$

has a singular point at $z = 0$. However, $\underset{z \to 0}{Lim}\left\{\frac{\sin z}{z}\right\} = 1$ and so the singularity at $z = 0$ is a removable singularity. We can see this from the Maclaurin series expansion of $f(z)$ where

$$f(z) = \frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

While we cannot substitute $z = 0$ into $f(z) = \frac{\sin z}{z}$, we can define $f(0) = 1$ in complete consistency with the series expansion. In this sense the singularity at $z = 0$ is removable by virtue of the fact that we can assign a value to $f(z)$ at the singularity which is consistent with the series expansion.

Move to the next frame

Circle of convergence

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When an expression is expanded in a Maclaurin series, the circle of convergence is always centred on the origin and the radius of convergence is determined by the location of the first singular point met as $|z|$ increases from $|z| = 0$. For example, the Maclaurin series expansion of $f(z) = \ln(1 + z)$ is

$$\ln(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots + \frac{(-1)^{n+1} z^n}{n} + \cdots$$

which is valid inside the circle of convergence $|z| = 1$. The first singular point met by this function as $|z|$ increases from zero is at $z = -1$, for at that point $\ln(1 + z)$ is not defined and the series

$$-1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n} - \cdots$$

diverges – it is the negative of the harmonic series. Hence the radius of convergence is 1. When $z = 1$, substitution into the series expansion gives

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots$$

▶

The right-hand side is the alternating harmonic series which we know converges by the *alternating sign test* which states that if the magnitude of the terms decreases and the signs alternate then the series converges. Now we know that it converges to $\ln 2$. Notice that the circle of convergence is identified by the location of the *first* singularity as $|z|$ increases from $|z| = 0$. This does not mean that the function is singular at all points on the circle of convergence.

There are times when it is desirable to have a series expansion of an expression that is singular at the origin. Because the Maclaurin expansion requires the function to be analytic everywhere within the circle of convergence which is centred on the origin, we cannot use that method. Fortunately, we do have a method of expanding a function about *any point* in the complex plane – this is Taylor's expansion.

Move to the next frame

Taylor's series

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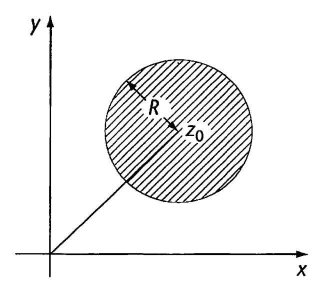
Provided $f(z)$ is analytic inside and on a simple closed curve c, the Taylor series expansion of $f(z)$ about the point z_0 which is interior to c is given as

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + \frac{(z-z_0)^2 f''(z_0)}{2!} + \cdots + \frac{(z-z_0)^n f^{(n)}(z_0)}{n!} + \cdots$$

where here, the point z_0 is the centre of the circle of convergence. The circle of convergence is given as $|z - z_0| = R$. That is $z - z_o = Re^{j\theta}$ or $z = z_0 + Re^{j\theta}$ where R is the radius of convergence.



Notice that Maclaurin's series is a special case of Taylor's series where $z_0 = 0$.

Example

Expand $f(z) = \dfrac{1}{z+1}$ in a Taylor series about the point $z = 1$ and find the values of z for which the expansion is valid.

The simplest way of doing this is to perform a coordinate transformation that moves the origin of the new coordinate to the point $z = 1$ and then derive the series about the new origin. To do this we define a new complex variable $u = z - 1$ so that $z = u + 1$ and so

$$\frac{1}{z+1} \text{ becomes } \frac{1}{u+2} = (2+u)^{-1} = \frac{1}{2}\left(1 + \frac{u}{2}\right)^{-1}.$$

▶

The expansion of this expression can now be derived using either Maclaurin or, as here, the binomial theorem to obtain

$$\frac{1}{u+2}=\frac{1}{2}\left(1+(-1)\frac{u}{2}+\frac{(-1)(-2)}{2!}\left(\frac{u}{2}\right)^2+\cdots\right)$$
$$=\frac{1}{2}-\frac{u}{4}+\frac{u^2}{8}-\frac{u^3}{16}+\cdots$$

Transforming back to the original variable z gives

$$\frac{1}{z+1}=\frac{1}{2}-\frac{z-1}{4}+\frac{(z-1)^2}{8}-\frac{(z-1)^3}{16}+\cdots$$

The circle of convergence is given by $\left|\frac{u}{2}\right|=1$, that is $\left|\frac{z-1}{2}\right|=1$ or $|z-1|=2$. Consequently, this series expansion is valid provided z is inside the circle defined by

$$z-1=2e^{j\theta} \text{ that is } z=1+2e^{j\theta}$$

By the same reasoning, the Taylor series expansion of $f(z)=\cos z$ about the point $z=\pi/3$ is

. . . .

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$$\boxed{\frac{1}{2}\left(1-\sqrt{3}(z-\pi/3)-\frac{(z-\pi/3)^2}{2!}+\sqrt{3}\frac{(z-\pi/3)^3}{3!}+\frac{(z-\pi/3)^4}{4!}-\cdots\right)}$$

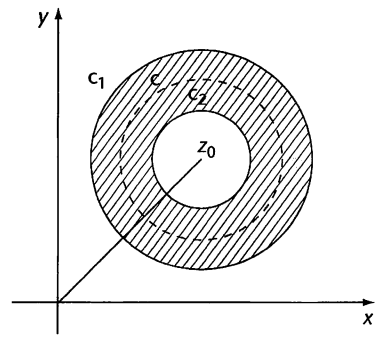
Because

If $u=z-\pi/3$ then

$$\begin{aligned}\cos z&=\cos(u+\pi/3)\\&=\cos u\cos\pi/3-\sin u\sin\pi/3\\&=\frac{1}{2}\left(\cos u-\sqrt{3}\sin u\right)\\&=\frac{1}{2}\left(\left[1-\frac{u^2}{2!}+\frac{u^4}{4!}-\cdots\right]-\sqrt{3}\left[u-\frac{u^3}{3!}+\frac{u^5}{5!}-\cdots\right]\right)\\&=\frac{1}{2}\left(1-\frac{u^2}{2!}+\frac{u^4}{4!}-\cdots-\sqrt{3}u+\sqrt{3}\frac{u^3}{3!}-\sqrt{3}\frac{u^5}{5!}-\cdots\right)\\&=\frac{1}{2}\left(1-\sqrt{3}u-\frac{u^2}{2!}+\sqrt{3}\frac{u^3}{3!}+\frac{u^4}{4!}-\sqrt{3}\frac{u^5}{5!}-\cdots\right)\\&=\frac{1}{2}\left(1-\sqrt{3}(z-\pi/3)-\frac{(z-\pi/3)^2}{2!}+\sqrt{3}\frac{(z-\pi/3)^3}{3!}\right.\\&\qquad\left.+\frac{(z-\pi/3)^4}{4!}-\cdots\right) \text{ for } z<\infty\end{aligned}$$

Laurent's series

Sometimes a valid series expansion of a function is required within a specific region of the complex plane that contains a singular point. In this case we cannot avoid the singular point as we did with Taylor's series by expanding about an alternative non-singular point, because then we move away from part of the specified region. To accommodate this case we can use the *Laurent series expansion* which provides a series expansion valid within an annular region *centred on the singular point.*



Let $f(z)$ be singular at $z = z_0$ and let c_1 and c_2 be two concentric circles centred on z_0. Then if $f(z)$ is analytic in the annular region between c_1 and c_2 and if c is any concentric circle lying within the annular region between c_1 and c_2 we can expand $f(z)$ as a Laurent series in the form

$$f(z) = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$
$$= \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$$

where $a_n = \dfrac{1}{2\pi j}\oint_c \dfrac{f(z)}{(z-z_0)^{n+1}}\,dz$

Example

Expand $\dfrac{e^{3z}}{(z-2)^4}$ in a Laurent series about the point $z = 2$ and determine the nature of the singularity at $z = 2$.

$f(z) = \dfrac{e^{3z}}{(z-2)^4}$ and $f'(z) = \dfrac{e^{3z}(3z-10)}{(z-2)^5}$ so $f(z)$ is analytic everywhere except at $z = 2$. The first thing we must do is to transform the coordinate system by shifting the origin to the point $z = 2$ by defining $u = z - 2$ so that $z = u + 2$. Then

$$\frac{e^{3z}}{(z-2)^4} = \frac{e^{3(u+2)}}{u^4} = e^6 \frac{e^{3u}}{u^4}.$$

▶

Now we can expand using the Maclaurin series expansion

$$=\frac{e^6}{u^4}\left\{1+3u+\frac{(3u)^2}{2!}+\frac{(3u)^3}{3!}+\frac{(3u)^4}{4!}+\frac{(3u)^5}{5!}+\cdots\right\}$$

$$=e^6\left\{\frac{1}{u^4}+\frac{3u}{u^4}+\frac{(3u)^2}{2!u^4}+\frac{(3u)^3}{3!u^4}+\frac{(3u)^4}{4!u^4}+\frac{(3u)^5}{5!u^4}+\cdots\right\}$$

$$=e^6\left\{\frac{1}{u^4}+\frac{3}{u^3}+\frac{9}{2u^2}+\frac{27}{6u}+\frac{81}{24}+\frac{243u}{120}+\cdots\right\}$$

$$=e^6\left\{\frac{1}{(z-2)^4}+\frac{3}{(z-2)^3}+\frac{9}{2(z-2)^2}+\frac{9}{2(z-2)}+\frac{27}{8}+\frac{81(z-2)}{40}+\cdots\right\}$$

This series converges for all finite z except $z=2$ at which point there is a pole of order 4.

The part of the Laurent series that contains negative powers of the variable is called the *principal part* of the series and the remaining terms constitute what is called the *analytic part* of the series. *If, in the principal part the highest power of $1/z$ is n, then the function possesses a pole of order n; and if the principal part contains an infinite number of terms, the function possesses an* **essential singularity**.

Now you try one

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The Laurent series expansion of $z^2\cos\frac{1}{z}$ about the point $z=0$

is .... valid for ....

at which point there is ....

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$$\boxed{\begin{array}{l} z^2-\frac{1}{2!}+\frac{1}{4!z^2}-\frac{1}{6!z^4}+\cdots \text{ valid for all } z\neq 0 \\ \text{at which point there is an essential singularity}\end{array}}$$

Because

$f(z)=z^2\cos\frac{1}{z}$ and $f'(z)=2z\cos\frac{1}{z}+\sin\frac{1}{z}$ and so $f(z)$ is analytic everywhere except at $z=0$. Expanding about $z=0$ gives

$$z^2\cos\frac{1}{z}=z^2\left(1-\frac{(1/z)^2}{2!}+\frac{(1/z)^4}{4!}-\frac{(1/z)^6}{6!}+\cdots\right)$$

$$=z^2-\frac{1}{2!}+\frac{1}{4!z^2}-\frac{1}{6!z^4}+\cdots$$

valid for all $z\neq 0$, at which point there is an essential singularity because there is an infinity of terms in the principal part of the series.

Try another. The Laurent series expansion of $\frac{z}{(z+2)(z+4)}$ valid for $2<|z|<4$ is ....

$$\boxed{\cdots+\frac{8}{z^4}-\frac{4}{z^3}+\frac{2}{z^2}-\frac{1}{z}+\frac{1}{2}-\frac{z}{8}+\frac{z^2}{32}-\frac{z^3}{128}+\cdots}$$

Because

$$\frac{z}{(z+2)(z+4)}=\frac{2}{z+4}-\frac{1}{z+2} \quad \text{(separating into partial fractions)}$$

If $|z|>2$ then we can write $\dfrac{1}{z+2}=\dfrac{1}{z(1+2/z)}=\dfrac{(1+2/z)^{-1}}{z}$

and because $|z|>2$, that is, $|2/z|<1$, we can now use the binomial theorem

$$\frac{1}{z+2}=\frac{1}{z(1+2/z)}=\frac{1}{z}\left\{1-\frac{2}{z}+\frac{4}{z^2}-\frac{8}{z^3}+\cdots\right\}=\frac{1}{z}-\frac{2}{z^2}+\frac{4}{z^3}-\frac{8}{z^4}+\cdots$$

and if $|z|<4$ then

$$\begin{aligned}\frac{2}{z+4}=\frac{1}{2(1+z/4)}&=\frac{1}{2}\left\{1-\frac{z}{4}+\frac{z^2}{16}-\frac{z^3}{64}+\cdots\right\}\\&=\frac{1}{2}-\frac{z}{8}+\frac{z^2}{32}-\frac{z^3}{128}+\cdots\end{aligned}$$

Note the expansion of $(1+z/4)^{-1}$ which is valid for $|z/4|<1$, that is $|z|<4$.

The first expansion for $|z|>2$ is still valid for $|z|<4$ since $4>2$ and the second expansion for $|z|<4$ is still valid for $|z|>2$ since $2<4$. Consequently, if $2<|z|<4$, then, by subracting the first series from the second

$$\begin{aligned}\frac{z}{(z+2)(z+4)}&=\frac{2}{z+4}-\frac{1}{z+2}\\&=\left\{\frac{1}{2}-\frac{z}{8}+\frac{z^2}{32}-\frac{z^3}{128}+\cdots\right\}-\left\{\frac{1}{z}-\frac{2}{z^2}+\frac{4}{z^3}-\frac{8}{z^4}+\cdots\right\}\\&=\cdots+\frac{8}{z^4}-\frac{4}{z^3}+\frac{2}{z^2}-\frac{1}{z}+\frac{1}{2}-\frac{z}{8}+\frac{z^2}{32}-\frac{z^3}{128}+\cdots\end{aligned}$$

Take care here! You may be tempted to think that this displays an essential singularity at $z=0$. This is not the case because the expansion is only valid inside the annular region $2<|z|<4$ centred on the origin. Consequently, the point $z=0$ is outside this region and the series expansion is invalid at that point.

The series expansion of the same function valid for $|z|<2$ is

....

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$$\boxed{\frac{z}{8}-\frac{3z^2}{32}+\frac{7z^3}{128}+\cdots}$$

Because

If $|z|<2$ then $\dfrac{1}{z+2}=\dfrac{1}{2(1+z/2)}=\dfrac{1}{2}\left\{1-\dfrac{z}{2}+\dfrac{z^2}{4}-\dfrac{z^3}{8}+\cdots\right\}$

$$=\frac{1}{2}-\frac{z}{4}+\frac{z^2}{8}-\frac{z^3}{16}+\cdots$$

We have already seen that if $|z|<4$ then

$$\frac{2}{z+4}=\frac{1}{2}-\frac{z}{8}+\frac{z^2}{32}-\frac{z^3}{128}+\cdots$$

This is still valid for $|z|<2$ since $2<4$. Conseqently, if $|z|<2$, then, by subtracting the first series from the second

$$\begin{aligned}\frac{z}{(z+2)(z+4)}&=\frac{2}{z+4}-\frac{1}{z+2}\\&=\left\{\frac{1}{2}-\frac{z}{8}+\frac{z^2}{32}-\frac{z^3}{128}+\cdots\right\}-\left\{\frac{1}{2}-\frac{z}{4}+\frac{z^2}{8}-\frac{z^3}{16}+\cdots\right\}\\&=\frac{z}{8}-\frac{3z^2}{32}+\frac{7z^3}{128}-\cdots\end{aligned}$$

Notice that for different regions of convergence we obtain different series expansions. Furthermore, each series expansion is unique within its own particular radius of convergence.

Try one more just to make sure that you can derive these expansions.

The Laurent series of $\dfrac{1-\cos(z-6)}{(z-6)^2}$ about the point $z=6$ is .... valid for .... at which point there is ....

20

$\dfrac{1}{2!}-\dfrac{(z-6)^2}{4!}+\dfrac{(z-6)^4}{6!}-\cdots$ valid for all $z\neq 6$
at which point there is a removable singularity

Because

If we let $u=z-6$ then

$$\begin{aligned}\frac{1-\cos(z-6)}{(z-6)^2}&=\frac{1-\cos u}{u^2}\\&=\frac{1}{u^2}\left\{1-\left(1-\frac{u^2}{2!}+\frac{u^4}{4!}-\frac{u^6}{6!}+\cdots\right)\right\}\\&=\frac{1}{2!}-\frac{u^2}{4!}+\frac{u^4}{6!}-\cdots\\&=\frac{1}{2!}-\frac{(z-6)^2}{4!}+\frac{(z-6)^4}{6!}-\cdots\end{aligned}$$

▶

This is valid for all finite values of $z \neq 6$ at which point there is a removable singularity which can be removed by defining $\dfrac{1-\cos(z-6)}{(z-6)^2}$ at $z=6$ as $\dfrac{1}{2!}$. Notice that here the principal part has no terms, so that the Laurent series is identical to the Taylor series.

Next frame

Residues

21

In the Laurent series

$$f(z) = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

the coefficient a_{-1} is referred to as the *residue* of $f(z)$ for reasons that will soon become apparent. Recall the integral in Frame 45 of Programme 21 which states that if the simple closed contour c has z_0 as an interior point, then

$$\oint_c \frac{dz}{(z-z_0)^n} = 2\pi j\delta_{n1}$$

where the Kronecker delta $\delta_{n1} = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } n \neq 1 \end{cases}$. Applying this fact to the Laurent series of $f(z)$ yields

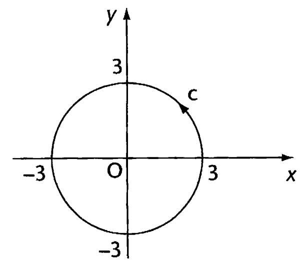
$$\begin{aligned}\oint_c f(z)\,dz &= \oint_c \left[\cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) \right.\\ &\qquad\qquad \left. + a_2(z-z_0)^2 + \cdots\right] dz \\ &= \cdots + \oint_c \frac{a_{-2}\,dz}{(z-z_0)^2} + \oint_c \frac{a_{-1}\,dz}{(z-z_0)} + \oint_c a_0\,dz \\ &\qquad + \oint_c a_1(z-z_0)\,dz + \oint_c a_2(z-z_0)^2 dz + \cdots \\ &= \cdots + 0 + 2\pi j a_{-1} + 0 + 0 + 0 + \cdots \\ &= 2\pi j a_{-1}\end{aligned}$$

That is, provided $f(z)$ is analytic at all points inside and on the simple closed contour c, apart from the single isolated singularity at z_0 which is interior to c, then

$$\oint_c f(z)\,dz = 2\pi j a_{-1}$$

Hence the name *residue* for a_{-1} because it is all that remains when the Laurent series is integrated term by term. This statement is called the **Residue theorem** and it has many far reaching consequences – we shall see some of these later. For now, just try an example.

▶



If c is a circle, centred on the origin and of radius 3, then

$$\oint_c \frac{z\,dz}{(z+2)(z+4)} = \ldots\ldots\ldots\ldots$$

22

$$\boxed{\oint_c \frac{z\,dz}{(z+2)(z+4)} = -2\pi j}$$

Because

The circle $|z| = 3$ lies within the annular region $2 < |z| < 4$ and we have already found the Laurent series for the integrand valid for $2 < |z| < 4$ in Frame 18, namely

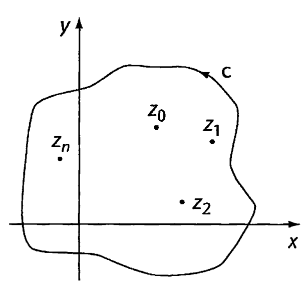
$$\frac{z}{(z+2)(z+4)} = \frac{2}{z+4} - \frac{1}{z+2}$$
$$= \cdots + \frac{8}{z^4} - \frac{4}{z^3} + \frac{2}{z^2} - \frac{1}{z} + \frac{1}{2} - \frac{z}{8} + \frac{z^2}{32} - \frac{z^3}{128} + \cdots$$

Here the residue is $a_{-1} = -1$ and so $\oint_c \frac{z\,dz}{(z+2)(z+4)} = 2\pi j(-1) = -2\pi j$

where c lies entirely within the region of convergence.

The Residue theorem extends to the case where the contour contains a finite number of singularities. If $f(z)$ is analytic inside and on the simple closed contour c except at the finite number of points z_0, z_1, $z_2, \ldots$, each with a Laurent series expansion and each with corresponding residues $\overset{(0)}{a}_{-1}, \overset{(1)}{a}_{-1}, \overset{(2)}{a}_{-1}, \ldots$ then

$$\oint_c f(z)\,dz = 2\pi j\left\{\overset{(0)}{a}_{-1} + \overset{(1)}{a}_{-1} + \overset{(2)}{a}_{-1}\right\} = 2\pi j\{\text{sum of residues inside c}\}$$



What could be more straightforward? Next frame

Calculating residues

23

When evaluating these integrals the major part of the exercise is to find the residues, and it would be very tedious if we had to find a Laurent series for each and every singularity. Fortunately there is a simpler method for poles. If $f(z)$ is analytic inside and on the simple closed contour c except at the interior point z_0 at which there is a pole of order n, then

$$a_{-1} = \underset{z\to z_0}{Lim}\left[\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left((z-z_0)^n f(z)\right)\right]$$

Example

Find the residues at all the poles of $f(z) = \dfrac{3z}{(z+2)^2(z^2-1)}$.

$f(z)$ has a pole of order 2 (a double pole) at $z=-2$ and two poles of order 1 (simple poles) at $z=\pm 1$.

At $z=-2$ the residue is $a_{-1} = \underset{z\to -2}{Lim}\left[\dfrac{1}{(2-1)!}\dfrac{d^{2-1}}{dz^{2-1}}\left((z+2)^2 f(z)\right)\right]$

$$= \underset{z\to -2}{Lim}\left[\frac{d}{dz}\left(\frac{3z}{z^2-1}\right)\right]$$

$$= \underset{z\to -2}{Lim}\left[\frac{3(z^2-1)-6z^2}{(z^2-1)^2}\right]$$

$$= \frac{3(4-1)-24}{(4-1)^2} = -\frac{5}{3}$$

At $z=1$ the residue is ....

24

$$\boxed{\frac{1}{6}}$$

Because

At $z=1$ the residue is $a_{-1} = \underset{z\to 1}{Lim}\left[\dfrac{1}{(1-1)!}\dfrac{d^{1-1}}{dz^{1-1}}\left((z-1)f(z)\right)\right]$

$$= \underset{z\to 1}{Lim}\left[\frac{d^0}{dz^0}\left(\frac{3z}{(z+2)^2(z+1)}\right)\right]$$

The zeroth derivative of an expression is the expression itself

$$= \underset{z\to 1}{Lim}\left[\frac{3z}{(z+2)^2(z+1)}\right]$$

$$= \frac{3}{(3)^2(2)} = \frac{1}{6}$$

At $z=-1$ the residue is ....

25

$$\boxed{\frac{3}{2}}$$

Because

$$\text{At } z=-1 \text{ the residue is } a_{-1} = \underset{z\to -1}{Lim}\left[\frac{1}{(1-1)!}\frac{d^{1-1}}{dz^{1-1}}((z+1)f(z))\right]$$

$$= \underset{z\to -1}{Lim}\left[\frac{d^0}{dz^0}\left(\frac{3z}{(z+2)^2(z-1)}\right)\right]$$

$$= \underset{z\to -1}{Lim}\left[\frac{3z}{(z+2)^2(z-1)}\right]$$

$$= \frac{-3}{(1)^2(-2)}$$

$$= \frac{3}{2}$$

Move to the next frame

Integrals of real functions

26

The Residue theorem can be very usefully employed to evaluate integrals of real functions that cannot be evaluated using the real calculus. Even when an integral is susceptible to evaluation by the real calculus, the use of the residue calculus can often save a great amount of effort. We shall look at three types of real integral and in each case we shall proceed by example.

Integrals of the form $\int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta$

Example

Evaluate $\displaystyle\int_0^{2\pi}\frac{1}{4\cos\theta - 5}\,d\theta.$

To evaluate this integral we make use of the exponential representation of a complex number of unit length, namely $z = e^{j\theta}$,and the exponential form of the trigonometric functions

$$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2} = \frac{z+z^{-1}}{2} \text{ and } \sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j} = \frac{z-z^{-1}}{2j},$$

and finally $dz = je^{j\theta}\,d\theta = jz\,d\theta$ so that $d\theta = dz/jz$

▶

Using these relations we can transform the real integral from 0 to 2π into a contour integral in the complex plane where the contour c is the *unit circle centred on the origin*. That is

$$\int_0^{2\pi} \frac{1}{4\cos\theta - 5}\,d\theta = \oint_c \frac{1}{4\dfrac{z+z^{-1}}{2} - 5} \times \frac{dz}{jz}$$

$$= -j\oint_c \frac{1}{2z^2 - 5z + 2}\,dz$$

$$= -j\oint_c \frac{1}{(2z-1)(z-2)}\,dz$$

The complex integrand has two simple poles, one at $z = \dfrac{1}{2}$ which is inside the contour c and another at $z = 2$ which is outside the contour c. Using the Residue theorem

$$-j\oint_c \frac{1}{(2z-1)(z-2)}\,dz = -j \times 2\pi j \times \{\text{residue at } z = 1/2\}$$

The residue at $z = 1/2$ is

$$\underset{z\to 1/2}{Lim}\left\{(z - 1/2)\frac{1}{(2z-1)(z-2)}\right\} = \underset{z\to 1/2}{Lim}\left\{\frac{1}{2(z-2)}\right\}$$

$$= -\frac{1}{3}$$

so that

$$\int_0^{2\pi} \frac{1}{4\cos\theta - 5}\,d\theta = -j\oint_c \frac{1}{(2z-1)(z-2)}\,dz$$

$$= -j \times 2\pi j \times \{\text{residue at } z = 1/2\}$$

$$= -2\pi/3$$

Now you try one

$$\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta} = \ldots\ldots\ldots\ldots$$

27

$$\boxed{\dfrac{2\pi}{\sqrt{3}}}$$

Because

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \oint_c \frac{dz/jz}{2+\dfrac{z+z^{-1}}{2}}$$ where c is the unit circle centred on the origin.

$$= -j\oint_c \frac{2\,dz}{z^2+4z+1}$$

$$= -j\oint_c \frac{2\,dz}{(z+2-\sqrt{3})(z+2+\sqrt{3})}$$

The integrand has two simple poles, one at $z=-2+\sqrt{3}$ which is inside c and another at $z=-2-\sqrt{3}$ which is outside c. Therefore

$$-j\oint_c \frac{2\,dz}{(z+2-\sqrt{3})(z+2+\sqrt{3})} = -j\times 2\pi j \times \left\{\text{residue at } z=-2+\sqrt{3}\right\}$$

The residue is

$$\underset{z\to -2+\sqrt{3}}{Lim}\left\{\left(z+2-\sqrt{3}\right)\frac{2}{(z+2-\sqrt{3})(z+2+\sqrt{3})}\right\}$$

$$= \underset{z\to -2+\sqrt{3}}{Lim}\left\{\frac{2}{(z+2+\sqrt{3})}\right\} = \frac{1}{\sqrt{3}}$$ and so

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = -j\oint_c \frac{2\,dz}{(z+2-\sqrt{3})(z+2+\sqrt{3})} = -j\times 2\pi j\times\frac{1}{\sqrt{3}}$$

$$= 2\pi\times\frac{1}{\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$

28

Integrals of the form $\int_{-\infty}^{\infty} F(x)\,dx$

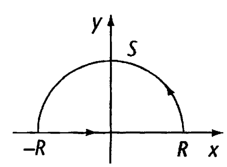
Example

Evaluate $\displaystyle\int_{-\infty}^{\infty}\frac{1}{1+x^4}\,dx$.

To evaluate this integral we must consider the integral $\displaystyle\oint_c \frac{1}{1+z^4}\,dz$ where c is the contour shown in the figure, so that

$$\oint_c \frac{1}{1+z^4}\,dz = \int_S \frac{dz}{1+z^4} + \int_{-R}^{R}\frac{dx}{1+x^4} = 2\pi j\{\text{sum of residues inside c}\}$$

Notice that along the real axis between $-R$ and R, $z=x$. Provided $R>1$ we can evaluate this integral using the Residue theorem. That is



$$\oint_c \frac{1}{1+z^4}\,dz = 2\pi j\times\{\text{sum of residues inside c}\}$$

29

$$\boxed{\frac{\pi}{\sqrt{2}}}$$

Because

The integrand $\dfrac{1}{1+z^4}$ possesses four simple poles at $z = e^{\pi j/4}$, $e^{3\pi j/4}$, $e^{5\pi j/4}$, $e^{7\pi j/4}$ of which only the first two are inside c.

$$\text{The residue at } z = e^{\pi j/4} \text{ is } \lim_{z \to e^{\pi j/4}} \left\{ \left(z - e^{\pi j/4}\right) \times \frac{1}{1+z^4} \right\}$$

$$= \lim_{z \to e^{\pi j/4}} \left\{ \frac{1}{4z^3} \right\} \text{ by L'Hôpital's rule}$$

$$= \frac{e^{-3\pi j/4}}{4}$$

$$\text{The residue at } z = e^{3\pi j/4} \text{ is } \lim_{z \to e^{3\pi j/4}} \left\{ \left(z - e^{3\pi j/4}\right) \times \frac{1}{1+z^4} \right\}$$

$$= \lim_{z \to e^{3\pi j/4}} \left\{ \frac{1}{4z^3} \right\} \text{ by L'Hôpital's rule}$$

$$= \frac{e^{-9\pi j/4}}{4}$$

$$= \frac{e^{-\pi j/4}}{4}$$

Therefore

$$\oint_c \frac{1}{1+z^4}\, dz = 2\pi j \times \left\{ \frac{1}{4}\left(e^{-3\pi j/4} + e^{-\pi j/4}\right) \right\}$$

Now $e^{-3\pi j/4} = \cos\dfrac{3\pi}{4} - j\sin\dfrac{3\pi}{4} = -\dfrac{1}{\sqrt{2}} - \dfrac{j}{\sqrt{2}}$ and

$$e^{-\pi j/4} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}} \text{ and so}$$

$$\oint_c \frac{1}{1+z^4}\, dz = 2\pi j \times \left\{ \frac{1}{4}\left(\frac{-2j}{\sqrt{2}}\right) \right\} = \frac{\pi}{\sqrt{2}}$$

We now look at the components of this integral in the next frame

30

We now recognise that

$$\oint_c \frac{1}{1+z^4}\, dz = \int_{-R}^{R} \frac{1}{1+x^4}\, dx + \int_S \frac{1}{1+z^4}\, dz$$

because $z = x$ along the real line.

Now we let R increase indefinitely and take limits, so that

$$\lim_{R\to\infty} \oint_c \frac{1}{1+z^4}\, dz = \int_{-\infty}^{\infty} \frac{1}{1+x^4}\, dx + \lim_{R\to\infty} \int_S \frac{1}{1+z^4}\, dz = \frac{\pi}{\sqrt{2}}$$

because the value of the contour integral is independent of the value of R. We shall now proceed to show that $\displaystyle\lim_{R\to\infty} \int_S \frac{1}{1+z^4}\, dz = 0$.

Writing $z = Re^{j\theta}$ so that, on S, $\mathrm{d}z = Re^{j\theta}\,\mathrm{d}\theta$, the limit of the integral becomes

$$\underset{R\to\infty}{Lim}\int_S \frac{Re^{j\theta}}{1 + R^4e^{j4\theta}}\,\mathrm{d}\theta = 0$$

Notice that the requirement that ensures that the integral along the semicircle vanishes in the limit is equivalent to the requirement that the degree of the denominator be at least two degrees higher than the numerator.

Now you try one.

$$\int_{-\infty}^{\infty}\frac{x^2\,\mathrm{d}x}{(x^2+1)^2} = \ldots\ldots\ldots\ldots$$

31

$$\boxed{\frac{\pi}{2}}$$

Because

Consider the integral $\oint_c \dfrac{z^2\,\mathrm{d}z}{(z^2+1)^2}$ where the contour c is the same semicircular contour as in the previous example. Here the integrand has two double poles at $z = j$ and $z = -j$ but only the pole at $z = j$ is inside the contour. The residue at $z = j$ is

$$\underset{z\to j}{Lim}\left\{\frac{\mathrm{d}}{\mathrm{d}z}(z-j)^2\frac{z^2}{(z-j)^2(z+j)^2}\right\} = \underset{z\to j}{Lim}\left\{\frac{2z(z+j)^2 - z^2 2(z+j)}{(z+j)^4}\right\}$$
$$= -\frac{j}{4}$$

Therefore

$$\oint_c \frac{z^2\,\mathrm{d}z}{(z^2+1)^2} = 2\pi j\left(-\frac{j}{4}\right) = \frac{\pi}{2}$$

Taking limits

$$\underset{R\to\infty}{Lim}\oint_c \frac{z^2\,\mathrm{d}z}{(z^2+1)^2} = \int_{-\infty}^{\infty}\frac{x^2\,\mathrm{d}x}{(x^2+1)^2} + \underset{R\to\infty}{Lim}\int_S \frac{z^2\,\mathrm{d}z}{(z^2+1)^2} = \frac{\pi}{2}$$

Where, in the second integral on the right-hand side, the degree of the denominator is two higher than the degree of the numerator, and so

$$\underset{R\to\infty}{Lim}\int_S \frac{z^2\,\mathrm{d}z}{(z^2+1)^2} = 0,\ \text{therefore}\ \int_{-\infty}^{\infty}\frac{x^2\,\mathrm{d}x}{(x^2+1)^2} = \frac{\pi}{2}$$

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Integrals of the form $\int_{-\infty}^{\infty} F(x)\begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix} dx$

These integrals are often referred to as Fourier integrals because of their appearances within Fourier analysis.

Example

Evaluate $\int_{-\infty}^{\infty} \frac{\cos kx}{a^2+x^2}\,dx$ where $a>0$ and $k>0$.

To evaluate this integral we consider the contour integral $\oint_c \frac{e^{jkz}}{a^2+z^2}\,dz$ where c is the semicircular contour of the previous problems and whose integrand possesses two simple poles at $z = aj$ and $z = -aj$ of which only the first is inside the contour. Consequently

$$\oint_c \frac{e^{jkz}}{a^2+z^2}\,dz = 2\pi j\{\text{residue at } z = aj\} = \ldots\ldots\ldots\ldots$$

33

$$\boxed{\frac{\pi e^{-ka}}{a}}$$

Because

The residue at $z = aj$ is

$$\underset{z\to aj}{Lim}\left\{(z-aj)\frac{e^{jkz}}{a^2+z^2}\right\} = \underset{z\to aj}{Lim}\left\{\frac{e^{jkz}}{z+aj}\right\} = \frac{e^{jk(aj)}}{2aj} = -\frac{je^{-ka}}{2a} \text{ and so}$$

$$\oint_c \frac{e^{jkz}}{a^2+z^2}\,dz = 2\pi j\left\{-\frac{je^{-ka}}{2a}\right\} = \frac{\pi e^{-ka}}{a}$$

Taking limits as $R\to\infty$

$$\underset{R\to\infty}{Lim}\oint_c \frac{e^{jkz}}{a^2+z^2}\,dz = \int_{-\infty}^{\infty}\frac{e^{jkz}}{a^2+z^2}\,dz + \underset{R\to\infty}{Lim}\int_S \frac{e^{jkz}}{a^2+z^2}\,dz = \frac{\pi e^{-ka}}{a}$$

In the second integral on the right-hand side, the degree of the denominator is two higher than the degree of the numerator, and so

$$\underset{R\to\infty}{Lim}\int_S \frac{e^{jkz}}{a^2+z^2}\,dz = 0, \text{ therefore } \int_{-\infty}^{\infty}\frac{e^{jkx}}{a^2+x^2}\,dx = \frac{\pi e^{-ka}}{a}. \text{ That is}$$

$$\int_{-\infty}^{\infty}\frac{\cos kx + j\sin kx}{a^2+x^2}\,dx = \frac{\pi e^{-ka}}{a} = 2\pi j\{\text{residue at } z = aj\}.$$

Consequently

$$\int_{-\infty}^{\infty}\frac{\cos kx}{a^2+x^2}\,dx = \frac{\pi e^{-ka}}{a} = -2\pi\,\text{Im}\{\text{residue at } z = aj\} \text{ and}$$

$$\int_{-\infty}^{\infty}\frac{\sin kx}{a^2+x^2}\,dx = 0 = 2\pi\,\text{Re}\{\text{residue at } z = aj\}$$

▶

Notice that e^{jkz} is easier to use than $\cos kx = (e^{jkx} + e^{-jkx})/2$, and it also gives the solution to the related integral with $\cos kx$ replaced with $\sin kx$.

Finally, to finish off the Programme, here is one for you to try.

$$\int_{-\infty}^{\infty} \frac{\cos \pi x}{x^2 + x + 1}\, dx = \dots\dots\dots\dots$$

34

$$\boxed{0}$$

Because

Consider $\displaystyle\oint_c \frac{e^{j\pi z}}{z^2 + z + 1}\, dz$ where c is the semicircular contour of the previous problem. The integrand is singular at the simple poles $z = (-1 \pm j\sqrt{3})/2$ where only $z = (-1 + j\sqrt{3})/2$ is inside the contour. The residue at $z = (-1 + j\sqrt{3})/2$ is then

$$\underset{z\to(-1+j\sqrt{3})/2}{Lim} \left\{ \left(z - \left[-1 + j\sqrt{3}\right]/2 \right) \frac{e^{j\pi z}}{z^2 + z + 1} \right\}$$

$$= \underset{z\to(-1+j\sqrt{3})/2}{Lim} \left\{ \frac{e^{j\pi z}}{z - \left[-1 - j\sqrt{3}\right]/2} \right\}$$

$$= \frac{e^{j\pi(-1+j\sqrt{3})/2}}{j\sqrt{3}}$$

$$= \frac{e^{-j\pi/2} e^{-\sqrt{3}\pi/2}}{j\sqrt{3}}$$

$$= -\frac{e^{-\sqrt{3}\pi/2}}{\sqrt{3}} \quad \text{since } e^{-j\pi/2} = -j$$

Therefore

$$\oint_c \frac{e^{j\pi z}}{z^2 + z + 1}\, dz = 2\pi j \left\{ \frac{e^{-\sqrt{3}\pi/2}}{\sqrt{3}} \right\} = -j\frac{2\pi e^{-\sqrt{3}\pi/2}}{\sqrt{3}}$$

that is

$$\oint_c \frac{e^{j\pi z}}{z^2 + z + 1}\, dz = \oint_c \frac{\cos \pi z + j \sin \pi z}{z^2 + z + 1}\, dz = -j\frac{2\pi e^{-\sqrt{3}\pi/2}}{\sqrt{3}}$$

and so

$$\oint_c \frac{\cos \pi z}{z^2 + z + 1}\, dz = 0 \text{ and } \oint_c \frac{\sin \pi z}{z^2 + z + 1}\, dz = -\frac{2\pi e^{-\sqrt{3}\pi/2}}{\sqrt{3}}$$

Note that, again, the contribution from the contour integral along the semicircle is zero.

The **Revision summary** now follows. Check it through in conjunction with the **Can You?** checklist before goint on to the **Test exercise**. The **Further problems** provide additional practice.

Revision summary 22

35

1 *Maclaurin series*

The Maclaurin series expansion of a function of a complex variable z is

$$f(z) = f(0) + zf'(0) + z^2\frac{f''(0)}{2!} + z^3\frac{f'''(0)}{3!} + \cdots$$

2 *Ratio test for convergence*

The ratio test for convergence of a series of terms of a complex variable

$$f(z) = a_0(z) + a_1(z) + a_2(z) + a_3(z) + \cdots + a_n(z) + \cdots$$

is that given

$$\underset{n\to\infty}{Lim}\left|\frac{a_{n+1}(z)}{a_n(z)}\right| = L$$

then if $L < 1$ the series converges and so the expansion is valid
$L > 1$ the series diverges and so the expansion is invalid
$L = 1$ the ratio test fails to give a conclusion.

3 *Radius and circle of convergence*

The radius of the circle within which a series expansion is valid is called the *radius of convergence* of the series and the circle is called the *circle of convergence*. The radius of convergence can be found using the ratio test for convergence.

4 *Singular points*

Any point at which $f(z)$ fails to be analytic, that is where the derivative does not exist, is called a *singular point.*

Poles

If $f(z)$ has a singular point at z_0 and for some natural number n

$$\underset{z\to z_0}{Lim}\left\{(z - z_0)^n f(z)\right\} = L \neq 0$$

then the singular point (also called a singularity) is called *a pole of order n.*

Removable singularity

If $f(z)$ has a singular point at z_0 but $\underset{z\to z_0}{Lim}\{f(z)\}$ exists then the singular point is called a *removable singularity.*

5 *Circle of convergence*

When an expression is expanded in a Maclaurin series, the *circle of convergence* is always centred on the origin and the *radius of convergence* is determined by the location of the first singular point met as z moves out from the origin.

6 *Taylor's series*

Provided $f(z)$ is analytic inside and on a simple closed curve c, the Taylor series expansion of $f(z)$ about a point z_0 which is interior to c is given as

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + \frac{(z - z_0)^2 f''(z_0)}{2!} + \cdots + \frac{(z - z_0)^n f^{(n)}(z_0)}{n!} + \cdots$$

where, here, the expansion is about the point z_0 which is the centre of the circle of convergence. The circle of convergence is given as $|z - z_0| = R$ where R is the radius of convergence. Maclaurin's series is a special case of Taylor's series where $z_0 = 0$.

7 *Laurent's series*

The *Laurent series expansion* provides a series expansion valid within an annular region *centred on the singular point.*

Let $f(z)$ be singular at $z = z_0$ and let c_1 and c_2 be two concentric circles centred on z_0. Then if $f(z)$ is analytic in the annular region between c_1 and c_2 and c is any concentric circle lying within the annular region between c_1 and c_2 we can expand $f(z)$ as a Laurent series in the form

$$f(z) = \cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{(z - z_0)} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

$$= \sum_{n \to -\infty}^{\infty} a_n(z - z_0)^n \text{ where } a_n = \frac{1}{2\pi j}\oint_c \frac{f(z)}{(z - z_0)^{n+1}}\,dz$$

8 *Residues*

In the Laurent series

$$f(z) = \cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{(z - z_0)} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

the coefficient a_{-1} is referred to as the *residue* of $f(z)$.

Residue theorem

Provided $f(z)$ is analytic at all points inside and on the simple closed contour c, apart from the single isolated singularity at z_0 which is interior to c, then

$$\oint_c f(z)\,dz = 2\pi j a_{-1}$$

9 The Residue theorem extends to the case where the contour contains a finite number of singularities. If $f(z)$ is analytic inside and on the simple closed contour c except at the finite number of points $z_0, z_1, z_2, \cdots$ each with a Laurent series expansion and each with corresponding residues $\overset{(0)}{a}_{-1}, \overset{(1)}{a}_{-1}, \overset{(2)}{a}_{-1}, \cdots$ then

$$\oint_c f(z)\,dz = 2\pi j\left\{\overset{(0)}{a}_{-1} + \overset{(1)}{a}_{-1} + \overset{(2)}{a}_{-1} + \cdots\right\}$$

▶

10 *Calculating residues*

$$a_{-1} = \underset{z \to z_0}{Lim}\left[\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}((z-z_0)^n f(z))\right]$$

11 *Real integrals*

The Residue theorem can be very usefully employed to evaluate integrals of real functions.

Integrals of the form $\int_0^{2\pi} F(\cos\theta,\ \sin\theta)\,d\theta$

Use $z = e^{j\theta}$ and the exponential form of the trigonometric functions $\cos\theta = \dfrac{e^{j\theta}+e^{-j\theta}}{2} = \dfrac{z+z^{-1}}{2}$, $\sin\theta = \dfrac{e^{j\theta}-e^{-j\theta}}{2j} = \dfrac{z-z^{-1}}{2j}$ and $dz = je^{j\theta}\,d\theta = jz\,d\theta$ so that $d\theta = dz/jz$. Convert the integral into a contour integral around the unit circle centred on the origin and use the Residue theorem.

Integrals of the form $\int_{-\infty}^{\infty} F(x)\,dx$ and $\int_{-\infty}^{\infty} F(x)\left\{\begin{matrix}\sin x\\ \cos x\end{matrix}\right.dx$

Consider integrals of the form $\oint_c F(z)\,dz$ and $\oint_c F(z)e^{jz}\,dz$ respectively, where the contour c is a semicircle with the diameter lying along the real axis. The principle is that the integral can be evaluated by the Residue theorem and then the contour can be expanded to cover the required extent of the real axis, the integration along the semicircle giving a zero contribution.

☑ Can You?

Checklist 22

36

Check this list before and after you try the end of Programme test

On a scale of 1 to 5 how confident are you that you can: Frames

- Expand a function of a complex variable about the origin in a Maclaurin series? 1 to 7

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Determine the circle and radius of convergence of a Maclaurin series expansion? 8 to 10

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Recognise singular points in the form of poles of order n, removable and essential singularities? 11

  Yes ☐ ☐ ☐ ☐ ☐ *No*

▶

- Expand a function of a complex variable about a point in the complex plane in a Taylor series, transforming the coordinates with a shift of origin? 12 to 14

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Expand a function of a complex variable about a singular point in a Laurent series? 15

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Recognise the principal and analytic parts of the Laurent series and link the form of the principal part to the type of singularity? 16 to 20

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Recognise the residue of a Laurent series and state the Residue theorem? 21 and 22

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Calculate the residues at the poles of an expression without resort to deriving the Laurent series? 23 to 25

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Evaluate certain types of real integrals using the Residue theorem? 26 to 34

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 22

37

1 Expand each of the following in a Maclaurin series and determine the radius and the circle of convergence in each case.

(a) $f(z) = e^z$

(b) $f(z) = \ln(1 + 4z)$.

2 Determine the location and nature of the singular points in each of the following.

(a) $f(z) = \dfrac{3z}{(z+1)^5}$

(b) $f(z) = z^{10}e^{1/z}$

(c) $f(z) = z\sin(1/z)$

(d) $f(z) = \dfrac{1 - \cos z}{z^2}$

3 Expand $f(z) = \sin z$ in a Taylor series about the point $z = \pi/4$ and determine the radius of convergence.

4 Expand each of the following in a Laurent series. In (a) and (c) determine the nature of the singularity from the principal part of the series.

(a) $f(z) = (5 - z)\cos\dfrac{1}{z+3}$ about the point $z = -3$

(b) $f(z) = \dfrac{2z}{(z+1)(z+3)}$ valid for $1 < |z| < 3$

(c) $f(z) = \dfrac{1}{z^3(z-2)^2}$ about the point $z = 2$.

5 Calculate the residues at each of the singularities of

$$f(z) = \frac{3z-1}{z^2(z+1)^2(z-1)}.$$

6 Evaluate each of the following integrals.

(a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5\cos\theta - 13}$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2 + x + 1}$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos 3x}{x^4 + 2x^2 + 1}\,dx$

Further problems 22

38

1 For each of the following find the Maclaurin series expansion and determine the radius of convergence.

(a) $\sinh z$

(b) $\tan z$

(c) $\ln\left(\dfrac{1+z}{1-z}\right)$

(d) a^z, where $a > 0$

(e) $\dfrac{15z^2}{(5-3z)^3}$.

2 By using the appropriate Maclaurin series expansions, show that

(a) $(\cos z)' = -\sin z$

(b) $\cos z = \dfrac{e^z + e^{-z}}{2}$

(c) $(e^z)' = e^z$.

▶

3 Given the series expansion for $(1+z)^{-1}$

(a) show by integration that this is compatible with the series expansion for $\ln(1+z)$

(b) by differentiation find $\sum_{n=1}^{\infty}(-1)^n n z^n$ and $\sum_{n=1}^{\infty}(-1)^n n^2 z^n$.

4 Use the ratio test to test each of the following for convergence.

(a) $\sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2}z^n$

(b) $\sum_{n=0}^{\infty}\frac{z^n}{1-3n}$

(c) $\sum_{n=0}^{\infty}\frac{n^2 z^n}{1-3n}$

(d) $\sum_{n=0}^{\infty}\frac{(\cos n\pi)z^n}{2n-1}$

(e) $\sum_{n=1}^{\infty}\frac{(-1)^n z^n}{(n+1)!}$

5 Find the Taylor series about the point indicated of each of the following.

(a) e^z about the point $z=2$

(b) $\cos z$ about the point $z=\pi/6$

(c) $(z-3)\sin(z+3)$ about the point $z=3$

(d) $(2z-5)^{-1}$ about the point $z=1/3$

(e) $(2z-5)^{-1}$ about the point $z=3$.

6 Find the series expansion of $z\ln z$ valid for $|z-1|<1$.

7 Find the circle of convergence of each of the following when expanded in a Taylor series about the point indicated.

(a) $e^{-z}\cos(z-2)$ about the point $z=1$

(b) $\frac{z^3}{(z^2+6)}$ about the point $z=0$

(c) $\frac{z-2}{(z-6)(z-4)}$ about the point $z=5$

(d) $\frac{z^2}{(e^z+1)}$ about the point $z=0$.

8 Locate and classify all of the singularities of each of the following.

(a) $\frac{(z-1)^3}{z^2(z^2-1)^2}$

(b) $z^{-2}e^{-1/z}$.

▶

9 Find the Laurent series about the point indicated of each of the following.

(a) $\dfrac{1}{z}\sin\left(\dfrac{1}{z}\right)$ about the point $z = 0$

(b) $\dfrac{1}{2z-3}$ about the point $z = 3/2$

(c) $\dfrac{z}{(z-2)(z-3)}$ about the point $z = 3$.

10 Find the Laurent series of $\dfrac{z-1}{(z+2)(z+5)}$ that is valid for

(a) $2 < |z| < 5$

(b) $|z| > 5$

(c) $|z| < 2$.

11 Evaluate each of the following integrals.

(a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\sin\theta}$

(b) $\displaystyle\int_0^{2\pi} \frac{d\theta}{\alpha+\beta\sin\theta}$ for $\alpha > |\beta|$

(c) $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+\alpha^2-2\alpha\cos\theta}$ where $0 < \alpha < 1$

(d) $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta d\theta}{5-4\cos\theta}$

(e) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta}$

(f) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+6x+13}$

(g) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2dx}{x^4+6x^2+13}$

(h) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2dx}{(x^2+4)^2}$

(i) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2+x+1}{x^4+x^2+1}dx$

(j) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^6+1}$

(k) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\sin\pi x\,dx}{x^4+6x^2+13}$

(l) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin\pi x}{x^4+x^2+1}dx$.